REMARKS ON THE PAPER BY Ia. I. SEKERZH-ZENKOVICH ENTITLED "ON THE THEORY OF THE BORDA MOUTHPIECE FOR A GAS"

(ZAMECHANIIA K RABOTE Ia.I. SEKERZH-ZEN'KOVICHA "K TEORII NASADKA BORDA DLIA GAZA")

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Chaplygin investigated the problem of gas efflux at subsonic velocity from an infinite vessel when the walls make an angle $q\pi$ between each other [1].

In [2] the special case of Chaplygin's problem is reviewed when $q = 2$, and the author maintains that in this particular case it is not possible to use Chaplygin's results.

It is shown in the present paper that Chaplygin's method and his results are in fact applicable in this case.

1. A plane gas stream, symmetrical with respect to the axis, flows from an infinite vessel through a Borda mouthpiece. We follow [2] in assuming the flow axis to be the intersection between the plane of symmetry and the flow plane. Introducing rectangular coordinate axes, we locate the x-axis to coincide with the mouthpiece axis and be directed into the vessel, the y-axis being directed vertically upwards. The origin is located where the vertical intersects the x-axis.

The notation of [2] is used: **V** is the stream velocity vector with components u and v along the axes; $\phi(x, y)$ and $\psi(x, y)$ are the velocity potential and stream function, respectively; κ is the ratio of specific heats of the gas; k is a coefficient in the adiabatic equation; θ is the angle between the velocity vector **V** and the x-axis.

$$\rho_0 = \rho|_{V=0} = \text{const}, \qquad \beta = \frac{1}{\varkappa - 1}, \qquad \tau = \frac{V^2}{2\alpha}, \quad \alpha = \frac{k\varkappa\rho_0^{\varkappa-1}}{\varkappa - 1}$$

The Chaplygin system of equations has the form

$$\frac{\partial\psi}{\partial y} = (1-\tau)^\beta \frac{\partial\varphi}{\partial x}, \qquad \frac{\partial\psi}{\partial x} = -(1-\tau)^\beta \frac{\partial\varphi}{\partial y} \tag{1.1}$$

or, in the hodograph plane

$$\frac{\partial\varphi}{\partial\theta}=\frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}, \qquad \frac{\partial\varphi}{\partial\tau}=-\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial\psi}{\partial\theta} \tag{1.2}$$

From Equation (1.2) Chaplygin derives an expression for the stream function in the form [1]

$$\lambda\psi=A+B\theta+\sum_{m=1}^{\infty}B_m\left(\frac{\tau}{\tau_0}\right)^m\frac{y_m(\tau)}{y_m(\tau_0)}\sin(2m\theta+\theta m) \tag{1.3}$$

where λ, A, B are constant, B_m is a numerical coefficient, $y_m(\tau)=y_{1/2n}(\tau)$ is the hypergeometric function studied by Chaplygin.

If we put

$$\lambda=\frac{\pi}{2Q}, \qquad \theta_m=-n\theta, \qquad A=-\frac{\pi}{2}, \qquad B=-\frac{1}{2} \tag{1.4}$$

and determine from the expansion

$$\theta=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\theta \qquad (-\pi<\theta<\pi)$$

the value of coefficient B_m, we obtain the result of [3] directly from (1.3):

$$\psi=-\frac{Q}{\pi}(\theta+\pi)+\frac{2Q}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{T_n(\tau)}{T_n(\tau_0)}\sin n\theta \tag{1.5}$$

where $T_n(\tau)=T_{2m}(\tau)=\tau^m y_m(\tau)$ is a regular particular solution of the equation for $\tau=0$:

$$\tau^2(1-\tau)\frac{d^2T}{d\tau^2}+\tau[1+(\beta-1)\tau]\frac{dT}{d\tau}-m^2[1-(2\beta+1)\tau]T=0 \tag{1.6}$$

Thus there is no need to solve the problem in order to find ψ by any other method, and it also becomes unnecessary to prove the possibility of double term-by-term differentiation of the series in (1.5) in τ and θ.

2. In [2] a formula was obtained for the contraction of a stream through a Borda mouthpiece in the form

$$\frac{h}{H}=\frac{T_1(\tau_0)}{2\tau_0T_1'(\tau_0)+T_1(\tau_0)} \tag{2.1}$$

In this expression H is the width of the mouthpiece, h is the stream width at infinity

$$T_1(\tau)=\sqrt{\tau}F(1,-\beta;2;\tau), \qquad T_1'(\tau_0)=\left(\frac{dT_1}{d\tau}\right)_{\tau=\tau_0}$$

$$F-\beta;2;\tau)=1+\sum^{\infty}\frac{(-\beta)(-\beta+1)\ldots(-\beta+m+1)}{(m+1)}\tau^m \tag{2.2}$$

Kibel [3] derived a general formula for the contraction of the stream in the form

$$\frac{h}{H} \quad \left(1 - \frac{8}{\pi q^2} \sin \frac{\pi q}{2} \sum_{m=1}^{\infty} (-1)^m \left[1 + \frac{\tau_0}{m} \frac{y'_{2m}(\tau_0)}{y_{2m}(\tau_0)}\right] \frac{m}{4m^2 / q^2 - 1}\right)^{-1} \tag{2.3}$$

With $q = 2$ in (2.3) we have indeterminancy, which is in fact easy to resolve. As a result we arrive at Formula (2.1) - a result from [2].

It should, additionally, be observed that the function $T_1(r)$ is an elementary one. This circumstance was apparently originally pointed out by Lighthill [4]:

$$T_1(\tau) = \frac{1}{(\beta + 1)\sqrt{\tau}} [1 - (1 - \tau)^{\beta+1}] \tag{2.4}$$

Thus, Formula (2.1) is considerably simplified:

$$\frac{h}{H} = \frac{1 - (1 - \tau_0)^{\beta+1}}{2(\beta + 1)\tau \quad (1 - \tau_0)^{\beta}} \tag{2.5}$$

The results of [2] follow from Formula (2.5) as a special case for $\beta = 2$ and $\beta = 3$. The necessity of tabulating the degree of contraction no longer exists, because (2.5) is obtained in closed form.

Parametric equations of the free stream profile follow from Chaplygin's results too, and the equation for y is given in [1] ($q = 1$).

It should be observed, finally, that if the quantities entering Formula (2.5) are expressed in terms of pressure (for instance, with the St. Venant and Wenzel formula) we arrivê at the formula for stream contraction given by S.A. Khristianovich

$$\frac{h}{H} = \frac{\varkappa - 1}{2\varkappa} \left(\frac{p_0}{p} - 1\right) \left[\left(\frac{p_0}{p}\right)^{\frac{\varkappa - 1}{\varkappa}} - 1\right]^{-1} \tag{2.6}$$

It is evident from Formula (2.3) that this elementary expression is valid only for $q = 2$.

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